Technical Notes and Correspondence

High-Frequency Nonlinear Vibrational Control

B. Shapiro and B. T. Zinn

***Abstract*—This paper discusses the feasibility of high-frequency nonlinear vibrational control. Such control has the advantage that it does not require state measurement and processing capabilities that are required in conventional feedback control. Bellman *et al.* [1] investigated nonlinear systems controlled by linear vibrational controllers and proved that vibrational control is not feasible if the Jacobian matrix has a positive trace. This paper extends previous work to include nonlinear vibrational controllers. A stability criteria is derived for nonlinear systems with nonlinear controllers, and it is shown that a nonlinear vibrational controller can stabilize a system even if the Jacobian matrix has a positive trace.**

***Index Terms*— Method of averaging, naturally occurring feedback, nonlinear control, open loop, vibrations.**

I. Introduction

This paper discusses the feasibility of applying open-loop control in the form of high-frequency vibrational control to engineering systems. Such control may be applied in cases where closed-loop control is impractical and has the advantage that it does not require costly sensing and computing capabilities. Vibrational control is applied by oscillating an accessible system component at low amplitude and high frequency (relative to the natural frequency of the system). For example, an inverted pendulum can be stabilized by vertically oscillating the pendulum pin at a sufficiently high frequency and low amplitude. Let us examine the case of the pendulum in more detail. The vertically oscillated pendulum is described by the following nonlinear differential equation:

$$\dot{x}_1 = x_2 \tag{1}$$
$$\dot{x}_2 = C\sin(x_1) - Bx_2 + aw^2 D\sin(x_1)\sin(wt) \tag{2}$$

where x_1 is the angular displacement measured from the inverted equilibrium point, x_2 is the angular velocity, B, C, and D are positive physical constants, and a and w are the amplitude and frequency of the applied vibration, respectively. In this example, the control input is the applied vibration which is given by $a\sin(wt)$. Note that the amplitude and frequency of the control input are constant and, therefore, independent of the state of the system. Since there is no sensing or computation involved, this is a form of open-loop control. However, (2) involves a feedback-like term $w^2 D\sin(x_1)$ which occurs naturally as a result of the moment arm $\sin(x_1)$ between the vertically oscillating pendulum pin and the center of mass of the pendulum. Consequently, the feedback $w^2 D\sin(x_1)$ is *naturally occurring*, as shown in Fig. 1.

Manuscript received July 19, 1995. This work was supported by the Davis S. Lewis, Jr., Chair and AFOSR under Contract F49620-93-1-0177, Dr. M. Birkan, contract monitor.

B. Shapiro was with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332 USA. He is currently with Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125 USA (e-mail: bshapiro@indra.caltech.edu).

B. T. Zinn is with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332 USA.

Publisher Item Identifier S 0018-9286(97)00492-3.

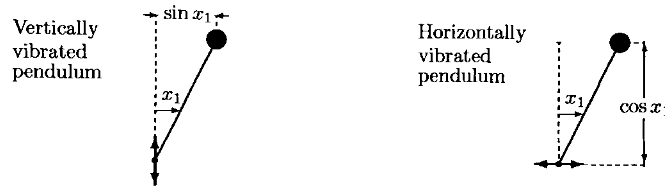

Fig. 1.

Since the naturally occurring feedback $w^2 D\sin(x_1)$ in (2) is of the same form as $C\sin(x_1)$, we can view this form of control as a variation of the parameter C; that is

$$\dot{x}_1 = x_2 \tag{3}$$
$$\dot{x}_2 = [C + aw^2 D\sin(wt)]\sin(x_1) - Bx_2. \tag{4}$$

Linearization of the above system yields

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ C + aw^2 D\sin(wt) & -B\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{5}$$

which is of the form

$$\dot{x} = [\mathcal{A} + \mathcal{B}(t)]x \tag{6}$$

where x is a vector, $\mathcal{A}$ is a constant matrix, and $\mathcal{B}(t)$ is a time-varying matrix. In the linear model (6), vibrational control appears as a variation of parameters, where the parameters of the matrix $\mathcal{A}$ are varied by $\mathcal{B}(t)$. This is the model investigated by Bellman *et al.* [1]. However, there is no reason to assume that vibrational control can always be viewed as a variation of parameters as in the above example. In fact, there are examples where the above model does not apply.

Consider the pendulum once again. Suppose we oscillate the pin of the pendulum horizontally instead of vertically, producing motions that are described by

$$\dot{x}_1 = x_2 \tag{7}$$
$$\dot{x}_2 = C\sin(x_1) - Bx_2 + aw^2 D\cos(x_1)\sin(wt). \tag{8}$$

Instead of the moment arm $\sin(x_1)$, we now have a moment arm $\cos(x_1)$, and the naturally occurring feedback is $w^2 D\cos(x_1)$. Linearization of this system of equations yields

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ C & -B\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ aw^2 D\sin(wt)\end{bmatrix} \tag{9}$$

which cannot be written in the form of (6). Consequently, we cannot view the above case as a variation of parameters.

The above example demonstrates that vibrating a system component does not always produce "variation of parameters" as in the vertically vibrated pendulum. Consequently, we adopt a more general approach that permits the analysis of problems, where a vibrated system component may result in nonlinear functions in the governing equations. Consider a nonlinear system

$$\dot{x} = f(x) \tag{10}$$

with an equilibrium point at the origin (i.e., $f(0) = 0$). Vibrational control is applied by oscillating a system component or process at high frequency and low amplitude. For instance, in the case of a jet engine, the air-throttle or amount of fuel injected might be vibrated. Let $h(wt) = \sin(wt)$ denote the applied high-frequency vibration. It is assumed that the vibration affects the system $f(x)$ through some naturally occurring feedback function $g(x, w, a)$, which depends on the vibrated component. The vibrationally controlled system is described by

$$\dot{x} = f(x) + h(wt)g(x, w, a). \tag{11}$$

For convenience, the amplitude of $h(wt)$ is taken to equal unity, and the amplitude of the applied vibration is accounted for by $g(x, w, a)$. In the case of the pendulum

$$f(x) = [x_2, C\sin(x_1) - Bx_2]^T \tag{12}$$

and $g(x, w, a) = [0, aw^2 D\sin(x_1)]^T$ for the vertically vibrated pin, or $g(x, w, a) = [0, aw^2 D\cos(x_1)]^T$ for the horizontally vibrated pin. We emphasize once again that $g(x, w, a)$ occurs naturally and is not measured or computed but is a result of the interaction between the system and vibrated component. Obviously, an oscillating fuel injection rate is not going to affect the jet engine in the same fashion as an oscillating throttle. Consequently, each actuation will be described by a different function $g(x, w, a)$. Since $g(x, w, a)$ depends on properties of the system (which are fixed) and the vibrated component, we can only control the choice of the component to oscillate and the frequency and amplitude of the vibration. This choice determines the form of $g(x, w, a)$, and since in certain cases there exist no $g(x, w, a)$ that will allow vibrational control, such control is not always feasible.

We now turn to the question of stability. Suppose the equilibrium point $x = 0$ of (10) is unstable, and that there exist one or more accessible system components or processes that can be vibrated, each associated with a function $g(x, w, a)$ that is known. The objective of the theory presented in this paper is to determine a stability criterion for (11). Consequently, if a certain $g(x, w, a)$ satisfies the derived stability criterion, then oscillation of the corresponding system component, with specific frequency w and amplitude a, will alter the stability of the system and result in vibrational control. Therefore, the developed criterion will determine if vibrational control is feasible for various accessible system components or processes in a given system.

Vibrational control has found various applications, including lasers [2] and particle beams [3]. Initial work on developing a general theory of vibrational control was carried out by Meerkov [4]. He discussed the effect of vibrational control upon stability, transient motion, and response of the controlled system. In subsequent publications, several specific nonlinear problems were discussed [5], but no general vibrational control was proposed. Such a theory was outlined by Bellman *et al.* [1], who presented criteria for the control of nonlinear systems by linear vibrational control. Further nonlinear results are discussed in [6], including conditions for and choice of stabilizing vibrations.

To discuss the results derived in [1], consider (11) and assume that the Jacobian matrix $\partial f(0)/\partial x = f'(0)$ of $f(x)$ in (11) has a positive trace. A classic theorem in linear algebra states that the trace of a matrix equals the sum of the real part of its eigenvalues (see for example [7, p. 251]). Consequently, if the trace is positive, then at least one of the eigenvalues must have a positive real part, and the equilibrium point is unstable. This does not imply, however, that if the trace is negative the equilibrium point is stable. A negative trace is a necessary but not a sufficient condition for stability.

Bellman *et al.* [1] only considered linear vibrational control, which limited the analysis to linear functions $g(x, a, w) = Mx$ in (11). They proved that if the Jacobian $f'(0)$ has a positive trace and $g(x, a, w)$ is linear, then vibrational control is not feasible, indicating that no matrix M can stabilize the system (11). In this paper, we consider a more general case of vibrational control via a nonlinear, slowly varying $g(x, a, w)$. In other words, we consider functions whose rate of change with respect to x is bounded (i.e., $\|\partial g/\partial x\| \le w\delta_1$). We show that in this case, vibrational control may be possible even if the trace of the Jacobian matrix is positive. Specifically, it will be shown that there exist nonlinear functions $g(x, a, w)$ that stabilize (11) even if its Jacobian $f'(0)$ has a positive trace.

The main point of this paper is that nonlinearities in $g(x, a, w)$ may not be negligible and can affect the stability of (11). This result is of practical importance for the following reason. In engineering, it is common practice to linearize a system before analyzing its stability. However, if a linear system is considered, then the Bellman *et al.* result indicates that vibrational control is not feasible when the Jacobian has a positive trace (note that positive traces occur in a wide variety of engineering systems, e.g., liquid rockets [8]). Most engineering systems are, however, nonlinear, and it is possible that nonlinearities in $g(x, a, w)$ may stabilize the system even if its Jacobian trace is positive. This implies that one should not discount vibrational control for systems that exhibit a positive trace. Instead, one should investigate the nonlinear functions $g(x, a, w)$ associated with vibrational open-loop control to determine if they satisfy the stability criteria derived in this paper. We also note that the theory presented in this paper agrees almost exactly with numerical solutions (see Section III-A).

II. General Derivation

Consider once again the nonlinear system

$$\dot{x} = f(x) + h(wt)g(x, w, a) \tag{13}$$

where $h(wt) = \sin(wt)$, $x \in \mathbb{R}^n$ is the state-space vector, and $x = 0$ is an equilibrium point of (10), which is not necessarily an equilibrium point of the forced system (13). It is assumed that $f(x)$ is three times continuously differentiable, and $g(x)$ is four times continuously differentiable.

We will show that the nonautonomous system (13) can be approximated by an autonomous system

$$\dot{y} = F(y). \tag{14}$$

This approximation means that there exists a function $u(t, y)$, which is small for all time, such that $x(t) = y(t) + u(t, y)$. Consequently, if $Y(t)$ is a solution of (14) and $X(t)$ is a solution of (13), then $X(t) - Y(t) = u[t, Y(t)]$ is small for all time t. Approximately, $Y(t)$ corresponds to the time average of $X(t)$ and it describes the slow response of the system, while $u[t, Y(t)]$ corresponds to the small amplitude high-frequency system oscillations excited by the small amplitude, high-frequency control input. In essence, there exist two time scales: a fast time scale corresponding to the high-frequency control input and the resulting high-frequency system response $u[t, Y(t)]$ and the slow time scale describing the time-averaged system response $Y(t)$. Since $Y(t)$ is a slow or averaged response, it is described by a time-averaged equation. In the case of vibrational control, the control input coupled with the system response $u[t, Y(t)]$ yields a nonzero average that can stabilize the system.

We will use the following notation. Since w and a are constant, we will express $g(x, w, a)$ as $g(x)$. Also, we define the Jacobian matrix $J = f'(0) = \partial f(0)/\partial x$ and let

$$p(x) = f(x) - Jx \tag{15}$$

$$\phi(wt) = -\varepsilon^2 Jg(0)\sin(wt) - \varepsilon g(0)\cos(wt) \tag{16}$$

where $\varepsilon = 1/w$, and $p(x)$ is the sum of all terms of second order and higher in the Taylor expansion of $f(x)$ around $x = 0$. Furthermore, we introduce the constant vector b

$$b = \left\{ \frac{1}{T} \int_0^T p[\phi(wt)]\, dt - \frac{\varepsilon^2 g'(0) J g(0)}{2} \right\} \tag{17}$$

where $T = 2\pi/w$ and $g'(0)$ is the Jacobian matrix of $g(x)$, and the constant matrix A

$$A = \left\{ J - \frac{\varepsilon^2}{2} \frac{\partial [g'(y) J g(y)]}{\partial y}(0) \right\} \tag{18}$$

where $g'(y)$ is the derivative of $g(y)$ evaluated at y, $\partial[g'(y)Jg(y)](0)/\partial y$ denotes the derivative of $g'(y)Jg(y)$ evaluated at zero. Finally, we let

$$\begin{aligned} \zeta = {} & \delta^2 + \varepsilon^2 \delta_0 \delta_1 + \delta \delta_1^2 + \frac{\delta_0 \delta_1^3}{\varepsilon} + \frac{\delta \delta_1^4}{\varepsilon} \\ & + \delta_0^3 + \delta_0^2 \delta_1 + \delta \delta_0^2 + \delta \delta_0 \delta_1 + \varepsilon^3 \delta_0^2 \end{aligned} \tag{19}$$

and denote a ball of radius δ centered at z as $B(z, \delta)$.

Theorem II.1: Consider the nonlinear system (13) and suppose that $f(0) = 0$, $\|g(0)\| \le w\delta_0$, and $\|g'(\xi)\| \le w\delta_1$ for all $\xi \in B(0, \delta)$. Then, for sufficiently small δ, δ_0, and δ_1, and sufficiently large w, there exists a function $u(t, y)$ that satisfies the following properties: $\|u(t, y)\| < 2(\delta_0 + \delta\delta_1)$ for all t and for all $y \in B(0, \delta)$; it is $2\pi/w$ periodic in t and for any y has zero mean value. Furthermore, for $x(t)$ governed by (13), $y(t) = x(t) - u(t, y)$ is governed by

$$\dot{y} = Ay + b + O(\zeta) \tag{20}$$

for all $y \in B(0, \delta)$ and b, A, and ζ defined in (17)–(19), respectively.

While a detailed proof of Theorem II.1 is given in the Appendix, an outline of the proof is provided below. A transformation $u(s, y)$ is constructed that satisfies the properties of the theorem. We then substitute the equation $y(t) = x(t) - u(t, y)$ into (13) and bound various terms so that we can rewrite (13) as the approximate system $\dot{y} = F(t, y)$. Next, we apply the method of averaging to derive the averaged equation $\dot{y} = F_{av}(y)$. Linearization of $\dot{y} = F_{av}(y)$ at the origin yields the result of the theorem.

The analysis in this paper includes Taylor terms up to second order in δ_0 and δ_1. Consequently, the resulting error ζ is of third order. If higher accuracy is desired, then more Taylor terms can be included, although more stringent smoothness constraints will be imposed because we will have to ensure that higher order derivatives exist for the functions $f(x)$ and $g(x)$. We note that for the examples considered, a second-order analysis is sufficient and is in excellent agreement with numerical integration results (see Example III-A).

A. Example: The Inverted Pendulum

Consider the vertically vibrated pendulum described by (1) and (2). These equations are of the form of (13). Since $g(0) = 0$, (16) implies that $\phi(wt) = 0$ and (15) shows that $p(0) = f(0) - 0 = 0$. Consequently, vector b defined in (17) equals zero. The matrix A is defined in (18) and can be expressed in the following form:

$$\begin{aligned} A &= \left\{ J - \frac{\varepsilon^2}{2} \frac{\partial [g'(y) J g(y)]}{\partial y}(0) \right\} \\ &= J - \frac{\varepsilon^2}{2} \frac{\partial}{\partial y} \left\{ \begin{bmatrix} 0 & 0 \\ a w^2 D \cos(y_1) & 0 \end{bmatrix} \right. \\ &\quad \cdot \left. \left. \begin{bmatrix} 0 & 1 \\ C & -B \end{bmatrix} \begin{bmatrix} 0 \\ a w^2 D \sin(y_1) \end{bmatrix} \right\} \right|_0 \\ &= \begin{bmatrix} 0 & 1 \\ C & -B \end{bmatrix} - \frac{\partial}{\partial y} \left. \begin{bmatrix} 0 \\ \dfrac{a^2 w^2 D^2 \cos(y_1) \sin(y_1)}{2} \end{bmatrix} \right|_0 \\ &= \begin{bmatrix} 0 & 1 \\ C - \dfrac{(awD)^2}{2} & -B \end{bmatrix}. \end{aligned} \tag{21}$$

Consequently, Theorem (II.1) indicates that the averaged behavior of the system is governed by

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ C - \dfrac{(awD)^2}{2} & -B \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{22}$$

which is in agreement with the result of [4]. Note that the term $C - (awD)^2/2$ is negative for sufficiently large a or w, indicating that the equilibrium point is asymptotically stable. We also note that even though the method in this paper is restricted to slowly varying $g(x)$, (i.e., $\|g'(x)\| \le w\delta_1 < w$), the above result is also valid for $\|g'(x)\| \not< w$. We impose the slowly varying restriction to permit inverting the matrix $[I + u_y]$ in (56). In the case of the pendulum, we can show that the matrix $[I + u_y]$ has an inverse even if $\|g'(x)\| \not< w$, which eliminates the slowly varying restriction.

III. Discussion of the Results

Theorem II.1 implies that vibrational control can result in an equilibrium shift. For such a shift to occur, the vector b defined in (17) has to be nonzero. Equations (15)–(17) imply that such an equilibrium shift can occur only if $g(0)$ is nonzero. In this case there are two possibilities. The first possibility is that the average of $p[\phi(wt)]$ is nonzero. Since $p(x)$ is defined in (15) as the nonlinear terms of $f(x)$, this implies that nonlinearities in $f(x)$ can cause an equilibrium shift. Such an equilibrium shift would be of order $O(\|\phi\|^2) = O(\delta_0^2)$. The second possibility is that the term $g'(0)Jg(0)$ is nonzero, indicating that the naturally occurring feedback function $g(x)$ can also cause an equilibrium shift. In this case, the equilibrium shift would be of order $O(\varepsilon^2 \|g'(0)Jg(0)\|) = O(\delta_0\delta_1)$. In either case, if the equilibrium shift is larger than δ, our analysis fails because we are forced outside the ball $B(0, \delta)$.

Theorem II.1 also yields a useful linear result. Consider a linear system of the form

$$\dot{x} = [J + \sin(wt) B] x \tag{23}$$

where $\|B\| < w\delta_1$. In this case, $g(x) = Bx$ and $g'(x) = B$. Therefore, $g(0) = 0$, and we can set $\delta_0 = 0$ with no loss of generality. Application of Theorem II.1 yields the averaged equation

$$\dot{y} = \left[J - \frac{\varepsilon^2 BJB}{2} \right] y + O\left[\delta \left(\delta + \delta_1^2 + \frac{\delta_1^4}{\varepsilon} \right) \right]. \tag{24}$$

However, the most interesting implication of Theorem II.1 is the following: the operator $g'(y)Jg(y)$ in (18) is a nonlinear operator in $g(y)$. Consequently, nonlinearities in $g(y)$ may result in linear terms in (20) and can influence local stability. This indicates that the local stability of the nonlinear system (11) is not the same as the stability of a corresponding linearized system. It is possible to show that the nonlinearities in $g(y)$ can alter the stability of a system with a positive Jacobian trace. Stabilization of a system with a positive trace is illustrated in the next example.

A. Example: A System with a Positive Jacobian Trace

In this example we consider a second-order system with a positive trace. Specifically, we consider the second-order system derived in [8] for the flow potential of a liquid rocket combustor

$$\ddot{x} + A_1 \dot{x} + A_0 x = 0 \tag{25}$$

where x is a nondimensional flow potential perturbation and t is a normalized time. In an unstable liquid rocket, unsteady combustion provides negative damping that drives the instability. Since the damping is determined by A_1, negative damping corresponds to a negative coefficient A_1. To illustrate the point, we let $A_1 = -0.2$ and $A_0 = 1$ and rewrite (25) as the following second-order system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -A_0 & -A_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0.2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{26}$$

The Jacobian matrix of (26) has a positive trace, indicating that the equilibrium point $x = 0$ is unstable.

Bellman *et al.* [1] prove that it is not possible to vibrationally control a system with a positive trace if the function $g(x)$ is linear. Consequently, postulate the existence of a nonlinear function $g(x)$

$$g(x) = \begin{bmatrix}0\\ \alpha + \beta x_1 x_2\end{bmatrix} \tag{27}$$

that describes the effect produced by forcing a system component. We stress once again that such a $g(x)$ would have to occur naturally. We will now show that if such a nonlinear $g(x)$ exists, it will stabilize the system (we do not claim that such a $g(x)$ is possible in rocket motors). A discussion of the reasoning for choosing the specific nonlinear $g(x)$ given in (27) is provided in Section C in the Appendix.

Given the above choice of $g(x)$, we write the forced equation as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -A_0 & -A_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \alpha + \beta x_1 x_2\end{bmatrix}\sin(wt). \tag{28}$$

Let $w = 70$, $\alpha = 15$, and $\beta = 200$. For these values, (28) becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0.2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 15 + 200 x_1 x_2\end{bmatrix}\sin(wt). \tag{29}$$

It follows from (27) that $\|g(0)\| \le \alpha \le w\delta_0$ for $\delta_0 \approx 0.22$. Similarly, $\|g'(x)\| \le \beta(x_1 + x_2) < 2\beta\delta < w\delta_1$ for $\delta_1 \approx (5.72)\delta$. Consequently, both δ_0 and δ_1 are sufficiently small, and we can apply Theorem II.1.

We need to calculate the vector b and the matrix A defined in (17) and (18), respectively. Notice that $p(x)$, defined in (15), is zero because the system (26) is linear. Consequently, (17) yields

$$b = -\frac{\varepsilon^2 g'(0) J g(0)}{2}. \tag{30}$$

However

$$g'(x) = \begin{bmatrix}0 & 0\\ \beta x_2 & \beta x_1\end{bmatrix} \tag{31}$$

indicating that $g'(0) = 0$, which implies $b = 0$ and that there is no equilibrium shift.

Equation (18) yields the matrix A

$$\begin{aligned} A &= \left[J - \frac{\varepsilon^2}{2}\frac{\partial[g'(y)Jg(y)]}{\partial y}(0)\right]\\ &= J - \frac{\varepsilon^2}{2}\frac{\partial}{\partial y}\\ &\quad \times \left\{\begin{bmatrix}0 & 0\\ \beta y_2 & \beta y_1\end{bmatrix}\begin{bmatrix}0 & 1\\ -A_0 & -A_1\end{bmatrix}\begin{bmatrix}0\\ \alpha + \beta y_1 y_2\end{bmatrix}\right\}\Bigg|_0\\ &= J - \frac{\varepsilon^2}{2}\frac{\partial}{\partial y}\begin{bmatrix}0\\ \alpha\beta y_2 - A_1\alpha\beta y_1 + \beta^2 y_1 y_2^2 - A_1\beta^2 y_1^2 y_2\end{bmatrix}\Bigg|_0\\ &= \begin{bmatrix}0 & 1\\ -A_0 + \dfrac{\varepsilon^2 A_1\alpha\beta}{2} & -A_1 - \dfrac{\varepsilon^2\alpha\beta}{2}\end{bmatrix}. \end{aligned} \tag{32}$$

Consequently, Theorem II.1 implies that the averaged motion of the system is governed by

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -A_0 + \dfrac{\varepsilon^2 A_1\alpha\beta}{2} & -A_1 - \dfrac{\varepsilon^2\alpha\beta}{2}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}. \tag{33}$$

Substituting the numerical values for A_0, A_1, α, β, and $\varepsilon = 1/w$ yields

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1.061 & -0.106\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} \tag{34}$$

which is asymptotically stable.

Since the solution $X(t)$ of (28) is given by $X(t) = Y(t) + u[t, Y(t)]$, where $Y(t)$ is a solution of (34) and tends toward the origin as time tends to infinity, $X(t)$ must remain close to the origin for all time because $u[t, Y(t)]$ is small for all time. The construction of $u(t, y)$, as defined in the Appendix [see (39), (43), and (44)], implies that if $g(0) \neq 0$ then $u(t, y) \not\to 0$ as $y \to 0$. In this case $g(0) \neq 0$, indicating that $u[t, Y(t)]$ does not converge to zero as $Y(t)$ tends to zero. Consequently, $X(t)$ remains close to zero for all time but does not tend to zero as time goes to infinity. Strictly speaking, the equilibrium point $x = 0$ of (29) is not asymptotically stable; indeed $x = 0$ is not an equilibrium point but is the center of a small asymptotically stable limit cycle. This limit cycle is the asymptotically stable orbit $X(t) = u(t, 0) \neq 0$. We refer to $x = 0$ as a *slow equilibrium point* because $y = 0$ is an equilibrium point of the slow or time-averaged system (20), and we say that $x = 0$ is *slowly asymptotically stable* because the equilibrium point $y = 0$ of the slow system (20) is asymptotically stable. When we refer to *slow* equilibrium points or *slow* stability, we refer to the properties of the time-averaged system (20). The true dynamics are small oscillations about the slow or averaged dynamics and hence display the same qualitative behavior. From a practical point of view we have achieved our control objective to keep (13) in a small neighborhood of the origin. Therefore, if there exists an accessible component in a liquid rocket motor that can produce a naturally occurring feedback function $g(x) = [0, \alpha + \beta x_1 x_2]^T$, then we can achieve vibrational control by vibrating this component.

It is interesting and instructive to compare results obtained by this analysis with a numerical simulation. We can analytically solve the time-averaged (34) to derive the following analytic expression for $Y_1(t)$:

$$Y_1(t) = e^{-0.053t}[Y_1(0)\cos(1.03t) + Y_2(0)\sin(1.03t)] \tag{35}$$

where $Y_1(0)$ is the initial displacement and $Y_2(0)$ is the initial velocity. Fig. 2 compares $Y_1(t)$ of (35) with an $X_1(t)$ calculated by numerically solving (29). Since the initial conditions for the slow solution $Y(t)$ are not known, they are matched to the initial conditions shown by the numerical simulation. Fig. 2 shows that the slow equilibrium point $x = 0$ of the forced system (29) is indeed slowly asymptotically stable (i.e., $X_1(t)$ approaches a small asymptotically stable limit cycle) but is not asymptotically stable ($X_1(t) \not\to 0$). Furthermore, Fig. 2 shows excellent agreement between the behavior predicted by the developed theory and the numerical simulation.

IV. Conclusion

In this paper, we present a criterion for nonlinear vibrational open-loop control. Previous work that was restricted to linear control is extended to include analysis of nonlinear, vibrational control. It has been previously shown that linear vibrational control is not feasible if the Jacobian matrix has a positive trace. This paper demonstrates that nonlinear vibrational control is possible even if the trace of the Jacobian is positive. This result is significant because a large number of nonlinear engineering systems exhibit a positive Jacobian trace and yet may be stabilized by nonlinear, open-loop, vibrational control. Finally, it is shown that the theory developed in this paper is in excellent agreement with numerical results.

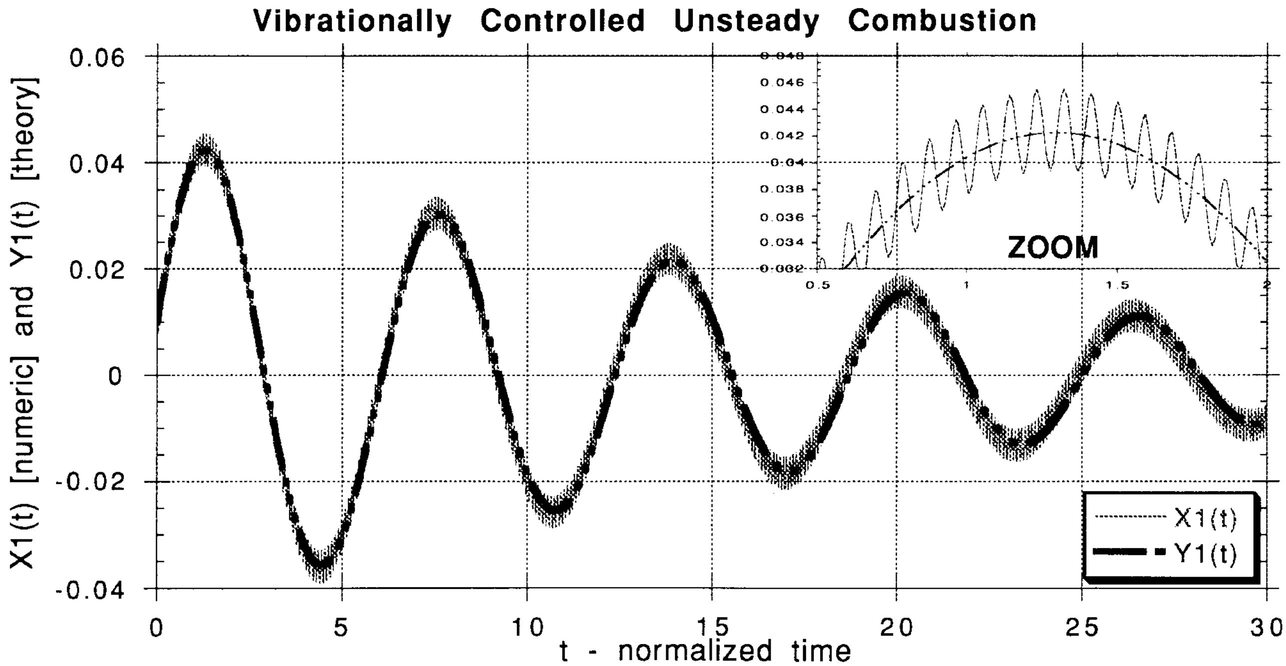

Fig. 2. Damping of a liquid rocket instability by high-frequency vibrational control.

Appendix

In this section, we prove Theorem II.1 and discuss the corresponding change of variables $x(t) = y(t) + u(t, y)$. We begin by assuming that the investigated system is described by

$$\dot{x} = f(x) + h(wt)g(x, w, a) \tag{36}$$

where $x \in \mathbb{R}^n$, $f \in C^3(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, $h(wt) = \sin(wt)$, $w \gg 1$, and $g \in C^4(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R}^n)$. We perform a local analysis that will be restricted to a ball of radius δ centered at the origin. In addition, since w and a are constant, we write $g(x, w, a)$ simply as $g(x)$ and impose the following smoothness constraints:

$$\begin{array}{rclcrcl} \|f'(0)\| & \leq & \sigma, & & 0 & \leq & \sigma \\ \|g(0)\| & \leq & w\delta_0, & & 0 & \leq & \delta_0 \\ \|g'(\xi)\| & \leq & w\delta_1, & & 0 & \leq & \delta_1 \end{array} \tag{37}$$

where $f'(x)$ denotes the derivative of f evaluated at x and $\xi \in B(0, \delta)$. To simplify the algebra, we introduce a fast time variable $s = wt$, define $\varepsilon = 1/w$, denote dx/ds as $\acute{x}$, and rewrite (36) in the fast time scale

$$\acute{x} = \varepsilon f(x) + \varepsilon h(s) g(x). \tag{38}$$

A. The Transformation

To prove Theorem II.1, we introduce the change of variables $x(s) = y(s) + u(s, y)$. Next, we define the function $u(s, y)$ and determine some of its properties; that is

$$u(s, y) = \alpha(y)\sin(s) + \beta(y)\cos(s) \tag{39}$$

where $\alpha, \beta \in C^4(\mathbb{R}^n, \mathbb{R}^n)$. The functions $\alpha(y)$ and $\beta(y)$ are chosen so that $u(s, y)$ satisfies the partial differential equation

$$u_s(s, y) = \varepsilon J u(s, y) + \varepsilon h(s) g(y) \tag{40}$$

where $J = f'(0)$ is the Jacobian matrix and the subscript s denotes a partial derivative with respect to s. Note that for any fixed y, the above equation is an ordinary differential equation in $u(\cdot, y)$. Substituting (39) into (40) and equating the coefficients of the sines and cosines yields

$$-\beta(y) - \varepsilon J\alpha(y) = \varepsilon g(y) \tag{41}$$
$$\alpha(y) - \varepsilon J\beta(y) = 0. \tag{42}$$

Solving (41) and (42) for $\alpha(y)$ and $\beta(y)$ yields

$$\alpha(y) = -\varepsilon^2[I + \varepsilon^2 J^2]^{-1} J g(y) \tag{43}$$
$$\beta(y) = -\varepsilon J\alpha(y) - \varepsilon g(y) \tag{44}$$

where the inverse matrix $[I + \varepsilon^2 J^2]^{-1}$ is well defined, provided ε is small enough to satisfy the inequality $\|\varepsilon^2 J^2\| < 1$.

To derive approximate equations for $\alpha(y)$ and $\beta(y)$ we need the following bound on $g(y)$:

$$\begin{aligned} \|g(y)\| &\leq \|g(0) + g(y) - g(0)\| \\ &\leq \|g(0)\| + \frac{\delta_1}{\varepsilon}\|y\| \\ &\leq \frac{\delta_0 + \delta_1\delta}{\varepsilon} \end{aligned} \tag{45}$$

which holds for all $y \in B(0, \delta)$. Next, we represent the inverse matrix $[I + \varepsilon^2 J^2]^{-1}$ as the geometric series

$$\begin{aligned} [I + \varepsilon^2 J^2]^{-1} &= I - \varepsilon^2 J^2 + \varepsilon^4 J^4 - \cdots \\ &= I + O(\varepsilon^2\sigma^2). \end{aligned} \tag{46}$$

Using (43)–(46) yields the following approximate expressions:

$$\alpha(y) = -\varepsilon^2 J g(y) + O(\varepsilon^3\delta_0 + \varepsilon^3\delta\delta_1) \tag{47}$$
$$\beta(y) = -\varepsilon g(y) + \varepsilon^3 J^2 g(y) + O(\varepsilon^4\delta_0 + \varepsilon^4\delta\delta_1). \tag{48}$$

To complete the discussion of the properties of $u(s, y)$, we need bounds on $u(s, y)$ and the partial derivative $u_y(s, y)$. We begin by bounding the inverse matrix $[I + \varepsilon^2 J^2]^{-1}$. Equation (46) implies

$$\begin{aligned} \|[I + \varepsilon^2 J^2]^{-1}\| &\leq \|I\| + \varepsilon^2\|J^2\| + \cdots \\ &\leq 1 + \varepsilon^2\sigma^2 + \cdots \\ &\leq \frac{1}{1 - \varepsilon^2\sigma^2}. \end{aligned} \tag{49}$$

It follows from (43)–(45) and (49) that

$$\|\alpha(y)\| \le \frac{\varepsilon\sigma(\delta_0+\delta\delta_1)}{1-\varepsilon^2\sigma^2} \tag{50}$$

$$\|\beta(y)\| \le \frac{\delta_0+\delta\delta_1}{1-\varepsilon^2\sigma^2} \tag{51}$$

for all $y \in B(0, \delta)$. To derive the desired bound on $u(s, y)$ we only need to note that (39) implies $\|u\| \le \|\alpha\| + \|\beta\|$, indicating that

$$\begin{aligned}\|u(s, y)\| &\le \frac{(1+\varepsilon\sigma)(\delta_0+\delta\delta_1)}{1-\varepsilon^2\sigma^2}\\ &< 2(\delta_0+\delta\delta_1)\\ &= O(\delta_0+\delta\delta_1)\end{aligned} \tag{52}$$

which holds for all s and sufficiently small ε. The bound on $u_y(s, y) = \alpha'(y)\sin(s) + \beta'(y)\cos(s)$ is also straightforward. Since (43) and (44) imply

$$\alpha'(y) = -\varepsilon^2[I+\varepsilon^2J^2]^{-1}Jg'(y) \tag{53}$$

$$\beta'(y) = -\varepsilon J\alpha'(y) - \varepsilon g'(y) \tag{54}$$

using (45), (49), (53), and (54), one obtains

$$\begin{aligned}\|u_y(s, y)\| &\le \frac{(1+\varepsilon\sigma)\delta_1}{1-\varepsilon^2\sigma^2}\\ &= O(\delta_1)\end{aligned} \tag{55}$$

which holds for all s.

B. Proof of Theorem II.1

We begin by noting that the transformation $u(s, y)$ constructed in the previous section satisfies the constraints outlined in the theorem. The transformation $x(s) = y(s) + u(s, y)$ implies $dx/ds = \acute{x} = \acute{y} + u_s + u_y\acute{y}$. Substituting this relationship into (38) yields

$$\begin{aligned}&[I+u_y(s, y)]\acute{y} + u_s(s, y)\\ &\quad = \varepsilon f(y+u) + \varepsilon h(s)g(y+u).\end{aligned} \tag{56}$$

Equation (55) implies $\|u_y(s, y)\| < 1$ for sufficiently small δ_1 for all $y \in B(0, \delta)$ and for all s. Consequently, the inverse matrix $[I + u_y(s, y)]^{-1}$ is well defined, and we can rewrite (56) as

$$\begin{aligned}\acute{y} = &[I+u_y(s, y)]^{-1}\\ &\cdot[\varepsilon f(y+u) + \varepsilon h(s)g(y+u) - u_s(s, y)].\end{aligned} \tag{57}$$

The following relationships will be used to simplify (57):

$$p(x) = f(x) - Jx \tag{58}$$

$$q(y, u) = g(y+u) - g(y) - g'(y)u. \tag{59}$$

where $p(x)$ is defined as before and $q(y, u)$ represents the sum of all terms of second order and higher in the Taylor expansion of $g(y+u)$ around $u = 0$. It follows that

$$\begin{aligned}p(0) &= 0\\ p'(0) &= 0\\ p(x) &= O(\|x\|^2)\end{aligned} \tag{60}$$

$$\begin{aligned}q(y, 0) &= 0\\ q_u(y, 0) &= 0\\ q(y, u) &= O(\|u\|^2).\end{aligned} \tag{61}$$

Using (58) and (59), we can rewrite (57) as

$$\begin{aligned}\acute{y} = &[I+u_y(s, y)]^{-1}[\varepsilon Jy + \varepsilon Ju(s, y) + \varepsilon p(y+u)\\ &+ \varepsilon h(s)g(y) + \varepsilon h(s)g'(y)u(s, y)\\ &+ \varepsilon h(s)q(y, u) - u_s(s, y)].\end{aligned} \tag{62}$$

Substituting (40) into (62) yields

$$\begin{aligned}\acute{y} = &[I+u_y(s, y)]^{-1}[\varepsilon Jy + \varepsilon p(y+u)\\ &+ \varepsilon h(s)g'(y)u(s, y) + \varepsilon h(s)q(y, u)].\end{aligned} \tag{63}$$

Approximating the inverse matrix $[I + u_y(s, y)]^{-1}$ as a two-term series with a second-order error

$$\begin{aligned}[I+u_y(s, y)]^{-1} &= I - u_y(s, y) + O(\|u_y\|^2)\\ &= I - u_y(s, y) + O(\delta_1^2)\end{aligned} \tag{64}$$

and substituting (64) into (63) yields

$$\begin{aligned}\acute{y} = &\varepsilon[I - u_y(s, y) + O(\delta_1^2)]\\ &\{Jy + h(s)g'(y)u(s, y) + p[y + u(s, y)]\\ &+ h(s)q[y, u(s, y)]\}\\ = &\varepsilon F(s, y).\end{aligned} \tag{65}$$

We are now in a position to apply the method of averaging. Since $F(s, y)$ is periodic in s with a period 2π, we can approximate the nonautonomous system $\acute{y} = \varepsilon F(s, y)$ as the autonomous averaged system $\acute{y} = \varepsilon F_{av}(y)$, where

$$F_{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} F(\tau, y)\,d\tau \tag{66}$$

(see [9, p. 412] for a discussion of averaging). Consequently, the averaged equation is given by

$$\begin{aligned}\acute{y} = &\frac{\varepsilon}{2\pi}\int_0^{2\pi}[I - u_y(\tau, y) + O(\delta_1^2)]\\ &\cdot\{Jy + h(\tau)g'(y)u(\tau, y) + p[y + u(\tau, y)]\\ &+ h(\tau)q[y, u(\tau, y)]\}\,d\tau.\end{aligned} \tag{67}$$

Expanding (67) yields

$$\begin{aligned}\acute{y} = &\frac{\varepsilon}{2\pi}\int_0^{2\pi}\Big\{Jy + h(\tau)g'(y)u(\tau, y)\\ &+ p[y+u(\tau, y)] + h(\tau)q[y, u(\tau, y)]\\ &- u_y(\tau, y)Jy - u_y(\tau, y)h(\tau)g'(y)u(\tau, y)\\ &- u_y(\tau, y)p[y+u(\tau, y)]\\ &- u_y(\tau, y)h(\tau)q[y, u(\tau, y)]\\ &+ O\left(\delta\delta_1^2 + \frac{\delta_0\delta_1^3}{\varepsilon} + \frac{\delta\delta_1^4}{\varepsilon} + \delta_0^2\delta_1^2\right)\Big\}\,d\tau.\end{aligned} \tag{68}$$

The terms $u_y(\tau, y)Jy$ and $u_y(\tau, y)h(\tau)g'(y)u(\tau, y)$ consist of an odd number of sinusoidal functions and thus average to zero. The term Jy is constant with respect to τ and can be taken outside the integral. Finally, since $h(s) = \sin(s)$ and $u(s, y) = \alpha(y)\sin(s) + \beta(y)\cos(s)$, averaging the term $h(\tau)g'(y)u(\tau, y)$ yields

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi} h(\tau)g'(y)u(\tau, y)\,d\tau\\ &\quad = \frac{1}{2\pi}\int_0^{2\pi}\sin^2(\tau)g'(y)\alpha(y)\\ &\qquad + \sin(\tau)\cos(\tau)g'(y)\beta(y)\,d\tau\\ &\quad = \frac{g'(y)\alpha(y)}{2}.\end{aligned} \tag{69}$$

Using the approximate expression (47) for $\alpha(y)$ in (69) lets us rewrite (68) as

$$\begin{aligned}\acute{y} = &\varepsilon Jy - \frac{\varepsilon^3}{2}g'(y)Jg(y) + \frac{\varepsilon}{2\pi}\int_0^{2\pi}\{p[y+u(\tau, y)]\\ &+ h(\tau)q[y, u(\tau, y)] - u_y(\tau, y)p[y+u(\tau, y)]\\ &- u_y(\tau, y)h(\tau)q[y, u(\tau, y)]\}\,d\tau\\ &+ \varepsilon O\left(\varepsilon^2\delta_0\delta_1 + \delta\delta_1^2 + \frac{\delta_0\delta_1^3}{\varepsilon} + \frac{\delta\delta_1^4}{\varepsilon} + \delta_0^2\delta_1^2\right).\end{aligned} \tag{70}$$

To complete the proof we have to bound the integral in (70). The bounds on $u_y(\tau, y)p[y + u(\tau, y)]$ and $u_y(\tau, y)h(\tau)q[y, u(\tau, y)]$ follow from (52), (55), (60), and (61); that is

$$\begin{aligned} u_y(\tau, y)p[y + u(\tau, y)] &= O(\|u_y\| \, \|y + u\|^2) \\ &= O(\delta^2\delta_1 + \delta\delta_0\delta_1 + \delta_0^2\delta_1) \qquad (71) \\ u_y(\tau, y)h(\tau)q[y, u(\tau, y)] &= O(\|u_y\| \, \|u\|^2) \\ &= O(\delta_0^2\delta_1 + \delta\delta_0\delta_1 + \delta^2\delta_1^3). \qquad (72) \end{aligned}$$

To get bounds on the remaining terms, $p[y + u(\tau, y)]$ and $h(\tau)q[y, u(\tau, y)]$, we will require the following notation. Denote the second-order Taylor expansion of $g(y + u)$ at $u = 0$ as

$$g(y + u) = g(y) + g'(y)u + \tfrac{1}{2}g''(y)\langle u, u\rangle + O(\|u\|^3) \quad (73)$$

where $\langle u, u\rangle$ denotes a tensor and $g''(y)$ is the corresponding three-dimensional array of coefficients evaluated at y. It follows that $q(y, u) = g''(y)\langle u, u\rangle/2 + O(\|u\|^3)$. Consequently, the average of $h(\tau)q(y, u(\tau, y))$ is written as

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} h(\tau)q[y, u(\tau, y)]\, d\tau \\ &\quad = \frac{1}{4\pi}\int_0^{2\pi} h(\tau)g''(y)\langle u(\tau, y), u(\tau, y)\rangle\, d\tau + O(\|u\|^3). \quad (74) \end{aligned}$$

Since each term of $h(\tau)\langle u(\tau, y), u(\tau, y)\rangle$ consists of an odd number of sinusoidal functions, the resulting average is zero. Hence, (74) is reduced to

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} h(\tau)q[y, u(\tau, y)]\, d\tau \\ &\quad = O(\delta_0^3 + \delta\delta_0^2\delta_1 + \delta^2\delta_0\delta_1^2 + \delta^3\delta_1^3). \quad (75) \end{aligned}$$

With the aid of bounds (71), (72), and (75), we can rewrite (70) as

$$\begin{aligned} \acute{y} = \varepsilon Jy - \frac{\varepsilon^3}{2}g'(y)Jg(y) + \frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau \\ + \varepsilon O\biggl(\varepsilon^2\delta_0\delta_1 + \delta\delta_1^2 + \frac{\delta_0\delta_1^3}{\varepsilon} \\ + \frac{\delta\delta_1^4}{\varepsilon} + \delta_0^3 + \delta\delta_0\delta_1 + \delta_0^2\delta_1\biggr). \quad (76) \end{aligned}$$

Equation (76) is of the form $\acute{y} = F(y) + \varepsilon O(\cdots)$ where

$$\begin{aligned} F(y) = \varepsilon Jy - \frac{\varepsilon^3}{2}g'(y)Jg(y) \\ + \frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau. \quad (77) \end{aligned}$$

Since we are concerned with local behavior at the origin, we linearize (76) about $y = 0$ to get

$$\acute{y} = F(0) + \left[\frac{\partial F}{\partial y}(0)\right]y + \varepsilon O(\delta^2 + \cdots). \quad (78)$$

Expanding the above yields

$$\begin{aligned} \acute{y} = \left\{\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[u(\tau, 0)]\, d\tau - \frac{\varepsilon^3 g'(0)Jg(0)}{2}\right\} \\ + \varepsilon Jy - \frac{\varepsilon^3}{2}\left\{\frac{\partial[g'(y)Jg(y)]}{\partial y}(0)\right\}y \\ + \left[\frac{\partial\left\{\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau\right\}}{\partial y}(0)\right]y \\ + \varepsilon O\biggl(\delta^2 + \varepsilon^2\delta_0\delta_1 + \delta\delta_1^2 + \frac{\delta_0\delta_1^3}{\varepsilon} \\ + \frac{\delta\delta_1^4}{\varepsilon} + \delta_0^3 + \delta\delta_0\delta_1 + \delta_0^2\delta_1\biggr). \quad (79) \end{aligned}$$

We now complete the proof by bounding the last term in (79). Since the derivative of $p(x)$ exists and is continuous by assumption, we can move the partial derivative $\partial/\partial y$ inside the integral to get

$$\begin{aligned} &\left[\frac{\partial\left\{\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau\right\}}{\partial y}(0)\right] \\ &\quad = \frac{\varepsilon}{2\pi}\int_0^{2\pi}\left[\frac{\partial\{p[y + u(\tau, y)]\}}{\partial y}(0)\right] d\tau \quad (80) \end{aligned}$$

where

$$\frac{\partial\{p[y + u(\tau, y)]\}}{\partial y}(0) = p'[u(\tau, 0)] + p'[u(\tau, 0)]u_y(\tau, 0). \quad (81)$$

Since $p(x) \in \mathbb{R}^n$, then $p'(a) \in \mathbb{R}^{n\times n}$ is a matrix-valued function. Letting $[M]_{ij}$ denote the ijth element of the matrix M and $\ell_{ij}(a) = [p'(a)]_{ij} \in \mathbb{R}$, using (81) lets us write the ijth term of (80) as

$$\begin{aligned} &\left[\frac{\partial\left\{\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau\right\}}{\partial y}(0)\right]_{ij} \\ &\quad = \frac{\varepsilon}{2\pi}\int_0^{2\pi} \ell_{ij}[u(\tau, 0)] \\ &\qquad + \langle \ell_{ik}[u(\tau, 0)][u_y(\tau, 0)]_{kj}\rangle\, d\tau \quad (82) \end{aligned}$$

where the tensor notation $\langle\ \rangle$ implies a summation over the index k. Expanding ℓ_{ij} and ℓ_{ik} as first- and zero-order Taylor series about the origin yields

$$\begin{aligned} &\left[\frac{\partial\left\{\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau\right\}}{\partial y}(0)\right]_{ij} \\ &\quad = \frac{\varepsilon}{2\pi}\int_0^{2\pi} \{\ell_{ij}(0) + \ell'_{ij}(0)u(\tau, 0) \\ &\qquad + \langle \ell_{ik}(0)[u_y(\tau, 0)]_{kj}\rangle\}\, d\tau \\ &\qquad + \varepsilon O(\|u\|^2 + \|u\|\,\|u_y\|). \quad (83) \end{aligned}$$

Equation (60) implies $\ell_{ij}(0) = 0$, and the averages of $\ell'_{ij}(0)u(\tau, 0)$ and $\ell_{ik}(0)[u_y(\tau, 0)]_{kj}$ are zero. Consequently

$$\begin{aligned} &\left[\frac{\partial\left\{\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[y + u(\tau, y)]\, d\tau\right\}}{\partial y}(0)\right]y \\ &\quad = \varepsilon O(\|y\|\,\|u\|^2 + \|y\|\,\|u\|\,\|u_y\|) \\ &\quad = O(\delta\delta_0^2 + \delta\delta_0\delta_1 + \delta^2\delta_1^2). \quad (84) \end{aligned}$$

Bound (84) allows us to rewrite (79) as

$$\begin{aligned} \acute{y} = \left[\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[u(\tau, 0)]\, d\tau - \frac{\varepsilon^3 g'(0)Jg(0)}{2}\right] \\ + \varepsilon Jy - \frac{\varepsilon^3}{2}\left[\frac{\partial[g'(y)Jg(y)]}{\partial y}(0)\right]y \\ + \varepsilon O\biggl(\delta^2 + \varepsilon^2\delta_0\delta_1 + \delta\delta_1^2 + \frac{\delta_0\delta_1^3}{\varepsilon} \\ + \frac{\delta\delta_1^4}{\varepsilon} + \delta_0^3 + \delta_0^2\delta_1 + \delta\delta_0^2 + \delta\delta_0\delta_1\biggr). \quad (85) \end{aligned}$$

According to definition (16)

$$\phi(\tau) = -\varepsilon^2 Jg(0)\sin(\tau) - \varepsilon g(0)\cos(\tau). \quad (86)$$

Then, (47) and (48) imply

$$\phi(\tau) = u(\tau, 0) + O(\varepsilon^3\delta_0 + \varepsilon^3\delta\delta_1). \tag{87}$$

Using (85) in (87) yields

$$\begin{aligned}\acute{y} &= \left[\frac{\varepsilon}{2\pi}\int_0^{2\pi} p[\phi(\tau)]\,d\tau - \frac{\varepsilon^3 g'(0)Jg(0)}{2}\right] \\ &\quad + \varepsilon Jy - \frac{\varepsilon^3}{2}\left[\frac{\partial[g'(y)Jg(y)]}{\partial y}(0)\right]y \\ &\quad + \varepsilon O\Bigg(\delta^2 + \varepsilon^2\delta_0\delta_1 + \delta\delta_1^2 + \frac{\delta_0\delta_1^3}{\varepsilon} \\ &\qquad + \frac{\delta\delta_1^4}{\varepsilon} + \delta_0^3 + \delta_0^2\delta_1 + \delta\delta_0^2 + \delta\delta_0\delta_1 + \varepsilon^3\delta_0^2\Bigg). \end{aligned} \tag{88}$$

Rewriting (88) in the original time scale t yields

$$\dot{y} = Ay + b + O(\zeta) \tag{89}$$

where A, b, and ζ are as defined in the theorem. □

C. Choice of $g(x)$ in a Positive Trace Example

In (27), we let $g(x) = [0,\, \alpha + \beta x_1 x_2]^T$. This hypothetical choice of $g(x)$ is not arbitrary. We know that the sign of A_1 creates an instability. Consequently, we wish to change the sign of this coefficient by applying vibrational control. Consider (20); if we denote the vector $g'(x)Jg(x)$ as $[G_1(x),\, G_2(x)]^T$, then the matrix A defined in (18) can be written as

$$\begin{aligned} A &= J - k\frac{\partial[G_1(x), G_2(x)]^T}{\partial x}(0) \\ &= J - k\begin{bmatrix} \dfrac{\partial G_1(0)}{\partial x_1} & \dfrac{\partial G_1(0)}{\partial x_2} \\ \dfrac{\partial G_2(0)}{\partial x_1} & \dfrac{\partial G_2(0)}{\partial x_2}\end{bmatrix} \end{aligned} \tag{90}$$

where k is a positive constant. For A to have a negative trace either $\partial G_1(0)/\partial x_1$ or $\partial G_2(0)/\partial x_2$ must be positive, or both. Consequently, letting $\partial G_2(0)/\partial x_2 = c$ be a positive quantity implies $G_2(x) = cx_2$. It follows that

$$\begin{aligned} G_2(x) &= g_2(x)\frac{\partial g_2}{\partial x_1} - g_1(x)\frac{\partial g_2}{\partial x_2} + 0.2g_2(x)\frac{\partial g_2}{\partial x_2} \\ &= cx_2. \end{aligned} \tag{91}$$

If we consider the first term only, we can set

$$g_2(x)\frac{\partial g_2}{\partial x_1} = cx_2. \tag{92}$$

Equation (92) is a partial differential equation in $g_2(x)$ which can be solved by the separation of variables. Unfortunately, the solution to (92) is $g_2(x) = \pm c_1\sqrt{|x_1 x_2|}$ which is singular at the origin and violates the assumption that $g(x)$ is continuously differentiable. Consequently, we let $g_2(x) = \alpha + \beta x_1 x_2$, approximating the square root dependence of $g_2(x)$ near the origin. If we now set $g_1(x) = 0$, then $g(x) = [0,\, \alpha + \beta x_1 x_2]^T$. It is noteworthy that the last term in (91) suggests that $g_2(x) = Kx_2$ might also be a viable feedback function. Such a choice requires, however, that $K > w$, which violates the assumption that $\|g'(x)\| \le w\delta_1 < w$.

ACKNOWLEDGMENT

This work benefited from many helpful discussions with Dr. J. Hale.

REFERENCES

[1] R. E. Bellman, J. Bentsman, and S. M. Meerkov, "Vibrational control of nonlinear systems: Vibrational stabilizability," *IEEE Trans. Automat. Contr.*, vol. 31, pp. 710–724, Aug. 1986.
[2] S. M. Meerkov and G. I. Shapiro, "Method of vibrational control in the problem of stabilization of ionization-thermal instability of a powerful continuous CO_2 laser," *Automat. Remote Contr.*, vol. 37, pp. 821–830, 1976.
[3] C. McGreavy and J. M. Thornton, "Stability studies of single catalyst particles," *Chemical Eng.*, vol. 1, pp. 296–301, 1970.
[4] S. M. Meerkov, "Principle of vibrational control: Theory and applications," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 755–762, Aug. 1980.
[5] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* New York: Springer-Verlag, 1983.
[6] J. Bentsman, "Vibrational control of a class of nonlinear systems by nonlinear multiplicative vibrations," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 711–716, Aug. 1987.
[7] G. Strang, *Linear Algebra and Its Applications.* New York: Harcourt Brace Jovanovich, 1988.
[8] E. Powell, "Nonlinear combustion instability in liquid propellant rocket engines," Ph.D. dissertation, Georgia Inst. Technol., Atlanta, GA, 1970.
[9] H. K. Khalil, *Nonlinear Systems.* New York: Macmillan, 1992.

On the Relation Between Local Controllability and Stabilizability for a Class of Nonlinear Systems

Sergej Čelikovský and Henk Nijmeijer

***Abstract*— The problem of local stabilizability of locally controllable nonlinear systems is considered. It is well known that, contrary to the linear case, local controllability does not necessarily imply stabilizability. A class of nonlinear systems for which local controllability implies local asymptotic stabilizability using continuous static-state feedback is described here, as for this class of systems the well-known Hermes controllability condition is necessary and sufficient for local controllability.**

***Index Terms*— Local controllability, nonlinear systems, stabilization, triangular form.**

I. INTRODUCTION

The aim of this contribution is to discuss local controllability of a class of nonlinear systems and its relation to stabilization by static-state feedback.

We study analytic single-input, continuous-time nonlinear control systems

$$\dot{x} = f(x) + ug(x) \tag{1}$$

with the state $x \in \mathbb{R}^n$ and the scalar input (or control) $u \in U$, a closed interval containing the origin. All considerations will be local in a neighborhood of an equilibrium point $x_E \in \mathbb{R}^n$, $f(x_E) = 0$ of

Manuscript received July 2, 1995; revised February 20, 1996. This work was supported in part by the Grant Agency of the Czech Republic under Grant 102/94/0053.

S. Čelikovský is with Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, 182 08 Prague, Czech Republic (e-mail: celikovs@utia.cas.cz).

H. Nijmeijer is with University of Twente, Department of Applied Mathematics, 7500 AE Enschede, The Netherlands.

Publisher Item Identifier S 0018-9286(97)01003-9.